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# One-way invisibility in isotropic dielectric optical media

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Optical materials with a distribution of loss and gain can be used to manipulate waves in fascinating ways, seemingly impossible with ordinary lossless materials. Some recent results have shown that (for planar media) if the spatial distributions of the real and imaginary parts of the permittivity are related to one another by the Kramers-Kronig relations, then reflection can be eliminated. Moreover, if an additional "cancellation condition" is satisfied, then a material can be made invisible for incidence from one side. Here, we give a simple demonstration of these results that should be accessible to undergraduates. In addition, we show how this simple method can be used to prove results about the reflection from permittivity profiles, without ever requiring an exact solution of the Helmholtz equation. 
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## I. INTRODUCTION

An optical wave propagating through an optical interface or in an inhomogeneous dielectric medium is usually partially reflected. Wave reflection is a fundamental phenomenon that arises whenever the refractive index changes on a spatial scale of the order of the optical wavelength, as discussed in many textbooks.[1] In many applications, reflection is an undesirable effect, and several methods have been developed to avoid wave reflection. For example, reflection from a sharp interface can be eliminated by applying an anti-reflection coating. There are also some well-known examples of non-reflecting dielectric profiles, such as the hyperbolic secant profile discussed in some textbooks,[2,3] belonging to the broad class of so-called reflectionless potentials originally introduced by Kay and Moses in 1956.[4] Other more advanced methods for designing reflectionless optical media are those based on transformation optics[5–9] and supersymmetry.[10,11]

Even though a dielectric profile can be designed to be *reflectionless* (such that it does not reflect any incident wave regardless of the incidence angle), generally it does not turn out to be *invisible* because the inhomogeneity introduces (in most cases) some shape distortion and spatial shift of the transmitted beam. An invisible scatter is, by definition, a localized inhomogeneity of the medium (which does not scatter any wave incident upon it) that appears to an outside observer as if there were no object at all present. Recently, great attention has been devoted to the possibility of designing isotropic inhomogeneous dielectric optical media (meaning no features of the magnetic permeability) possessing a complex refractive index profile (meaning loss and/or gain in amplitude during propagation) that appear to be invisible when probed from one side but not from the opposite side (*one-way invisibility*).[12–16]

An important step toward a general understanding of the reflection properties of optical waves in inhomogeneous media with a complex refractive index has been given in Ref. 16. In that work, the authors discovered the very intriguing property that any spatially inhomogeneous planar dielectric medium with a complex dielectric permittivity profile $\epsilon(x)$, such that its real and imaginary parts are related by Kramers-Kronig relations, does not scatter light when probed from one side. In such media, one-way invisibility is simply ensured when the so-called "cancellation condition" is satisfied.[17,18] In their original paper, Horsley *et al.* presented two different proofs of the above-mentioned property.[16] Subsequent works[17–19] re-considered the problem and presented different proofs, clarifying some subtleties that might arise in the formulation of the scattering problem. However, all previous demonstrations exploit some advanced tools of either complex function analysis or Green function theory that might not be accessible to a broad class of researchers and students in optics and photonics. It would be desirable to present the main results and proofs in a simple (yet rigorous) level.

In this article, we introduce at an elementary level the problem of wave reflection from an inhomogeneous planar dielectric medium and present a simple proof of the one-way invisibility property of a wide class of inhomogeneous isotropic optical media with a complex refractive index, expected to be accessible to a broad audience. The proof only involves elementary facts of scattering theory, and a simple relationship between the reflection and transmission spectra of partner dielectric permittivity profiles connected by a complex spatial displacement.[18,19] An example of one-way invisible permittivity profile, synthesized by the complex spatial displacement method, is also discussed.

## II. WAVE REFLECTION FROM AN INHOMOGENEOUS DIELECTRIC SLAB: BASIC EQUATIONS

Let us consider the electromagnetic scattering of a monochromatic optical wave at frequency $\omega$ from an inhomogeneous isotropic planar dielectric medium in the $xy$-plane as shown in Fig. 1. We indicate by $\epsilon = \epsilon(x) = n^2(x)$ the dielectric permittivity profile, where the real and imaginary parts of the refractive index $n(x)$ determine the local wave number and amplification/attenuation coefficient of the wave. The inhomogeneity is assumed to be localized at around $x = 0$, so that $\epsilon(x) \to \epsilon_b$ as $x \to \pm\infty$, where $\epsilon_b = n_b^2$ and $n_b$ is the refractive index of the substrate. If dissipation in the substrate can be neglected, $n_b$ and hence $\epsilon_b$ are real. For a monochromatic wave, we take the electric and magnetic fields in the form of spatially dependent amplitudes multiplied by a time dependent phase, so that $\mathcal{E}(x,z,t) = \mathbf{E}(x,y)e^{-i\omega t} + \text{c.c.}$ and $\mathcal{H}(x,z,t) = \mathbf{H}(x,y)e^{-i\omega t} + \text{c.c.}$, with invariance along the $z$-direction (c.c. here means complex conjugate). With this substitution, Maxwell's equations become the following coupled equations for the electric and magnetic field amplitudes $\mathbf{E}(x,y)$ and $\mathbf{H}(x,y)$:

$$\nabla \cdot (\epsilon \mathbf{E}) = 0, \tag{1}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{2}$$

$$\nabla \times \mathbf{E} = i\omega\mu_0 \mathbf{H}, \tag{3}$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon_0\epsilon(x)\mathbf{E}. \tag{4}$$

Because here the permittivity $\epsilon$ does not vary as a function of $z$, a plane wave propagating in the $xy$-plane and polarized along $\hat{\mathbf{z}}$ retains the same polarization everywhere. We focus our analysis on so-called TE-polarized waves, where the electric field $\mathbf{E} = E_z\hat{\mathbf{z}}$ oscillates in the $z$-direction while the magnetic vector $\mathbf{H}$ lies in the $xy$-plane (see Fig. 1). After writing the divergence and curl operators in Eqs. (1)–(4) in Cartesian coordinates and taking into account that only $E_z$, $H_x$, and $H_z$ are non-zero, Eqs. (3)–(4) reduce to

$$H_x = \frac{1}{i\mu_0\omega}\frac{\partial E_z}{\partial y}, \tag{5}$$

$$H_y = -\frac{1}{i\mu_0\omega}\frac{\partial E_z}{\partial x}, \tag{6}$$

and

$$E_z = \frac{i}{\omega\epsilon_0\epsilon(x)}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right). \tag{7}$$

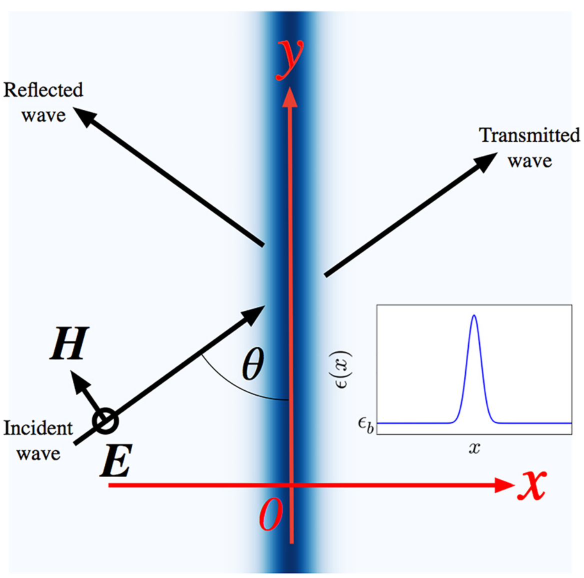


Fig. 1. Schematic diagram showing reflection and transmission of a TE-polarized optical wave in a planar dielectric medium with an inhomogeneous permittivity profile $\epsilon = \epsilon(x)$ (as an example here $\epsilon(x)$ is shown as a real-valued Gaussian profile). The inhomogeneity is localized near $x=0$, i.e., $\epsilon(x) \to \epsilon_b$ (real and positive) far from the interface region $x=0$.

Substitution of Eqs. (5) and (6) into Eq. (7) shows that the electric field amplitude $E_z$ evolves according to the Helmholtz equation

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_0^2\epsilon(x)E_z = 0, \tag{8}$$

where $k_0 = \omega/c$ is the wavenumber in vacuum. For the situation described above, the refractive index reaches some constant value as $|x| \to \infty$, and the relative dielectric permittivity $\epsilon(x)$ can be written in the form

$$\epsilon(x) = \epsilon_b + \beta(x), \tag{9}$$

where $\beta(x)$ vanishes as $|x| \to \infty$. Here, we imagine that while $\epsilon_b = n_b^2$ is real, $\beta(x)$ is some *complex* function of position, meaning the inhomogeneous planar medium may exhibit local loss and/or gain. As we shall see, there are an enormous number of possible reflectionless or invisible slabs of material when the loss or gain varies as a function of position. Meanwhile, in the absence of local gain and loss, it is extremely challenging to realize such effects.

Now consider an optical wave incident onto the inhomogeneous region of the dielectric from the left side (coming from $x \to -\infty$), at some incidence angle $\theta$ ($\theta \to 0$ for grazing incidence), as shown in Fig. 1. We look for a solution to Eq. (8) of the form

$$E_z(x,y) = \psi(x)e^{ik_y y}, \tag{10}$$

where $0 < k_y < k_0 n_b$. The incidence angle $\theta$ is given by $\theta = \tan^{-1}(k_x/k_y)$, where we have set $k_x = \sqrt{k_0^2\epsilon_b - k_y^2} > 0$. The amplitude $\psi(x)$ then satisfies a stationary one-dimensional Schrödinger-like wave equation

$$\hat{H}\psi(x) = E\psi(x), \tag{11}$$

where $E = k_x^2 = k_0^2\epsilon_b - k_y^2$ is the "energy" and $\hat{H} = -d^2/dx^2 + V(x)$ is the Hamiltonian with the *optical potential* $V(x)$ given by

$$V(x) = -k_0^2\beta(x) = -k_0^2[\epsilon(x) - \epsilon_b]. \tag{12}$$

At large distances from the origin, the solutions to Eq. (11) become sums of plane waves $e^{\pm ik_x x}$ (although this is only true for an optical potential that vanishes sufficiently fast as $x \to \pm\infty$, as described in Ref. 17). The amplitudes of these plane waves are determined by the transmission $t(k_x)$ and reflection $r^{(L)}(k_x)$ coefficients of the profile, with $(L)$ indicating that the waves are incident from the left-hand side, and on each side of the profile the wave takes the form

$$\psi(x) = \begin{cases} e^{ik_x x} + r^{(L)}(k_x)e^{-ik_x x} & x \to -\infty \\ t(k_x)e^{ik_x x} & x \to \infty. \end{cases} \tag{13}$$

Similarly, for right-side incidence, the wave takes the form

$$\psi(x) = \begin{cases} t(k_x)e^{-ik_x x} & x \to -\infty \\ e^{-ik_x x} + r^{(R)}(k_x)e^{ik_x x} & x \to \infty. \end{cases} \tag{14}$$

To find these reflection and transmission coefficients, one needs to solve Eq. (11) and take the limits $x \to \pm\infty$ of the solutions. There are not many cases where this can be done analytically, but we shall establish some rather general results below that do not require knowledge of the exact solution to the Schrödinger equation. Note that while the transmission coefficient $t(k_x)$ does not depend on the incidence side (which is an illustration of the phenomenon of reciprocity), the reflection coefficient *does*, and generally $r^{(L)}(k_x) \neq r^{(R)}(k_x)$.

In a medium with a real permittivity profile $\epsilon(x)$ one has, in addition, a conservation of energy flux that constrains the values of the reflection coefficients. This conservation law can be obtained by taking Eq. (8), multiplying it by $E_z^*$, and subtracting the complex conjugate of the resulting equation. This leads to

$$\nabla \cdot (\mathrm{Im}\left[E_z^* \nabla E_z\right]) = -k_0^2\,\mathrm{Im}[\epsilon(x)]|E_z|^2, \tag{15}$$

which, for fields of the form given in Eq. (10), implies that $\mathrm{Im}[E_z^* \partial E_z/\partial x]$ is independent of $x$ when $\mathrm{Im}[\epsilon] = 0$ throughout space. Substituting Eq. (13) or Eq. (14) into this conservation law on the two sides of the slab leads to $|r^{(L,R)}(k_x)|^2 + |t(k_x)|^2 = 1$, which in terms of individual photons, implies that an equal number are emitted from the slab as are shone onto it. For the lossless case, we can then see that reciprocity constrains the magnitudes of the two reflection coefficients to be equal, meaning $|r^{(L)}(k_x)|^2 = |r^{(R)}(k_x)|^2$. In general, these conditions do not hold in the cases examined in this paper because we assume $\mathrm{Im}[\epsilon] \neq 0$. This assumption allows us to find complex *one-way invisible* optical potentials $V(x)$. These potentials are such that the medium transmits without any phase shift, $t(k_x) = 1$, and also does not reflect waves incident from the left-hand side only. Thus, $r^{(L)}(k_x) = 0$ for any $k_x > 0$ or for any incidence angle $\theta$ (in general $r^{(R)}(k_x) \neq 0$).

## III. REFLECTION/TRANSMISSION COEFFICIENTS OF SPATIALLY DISPLACED OPTICAL POTENTIALS

In order to find these one-way invisible potentials, we shall present an interesting and very simple relation connecting the reflection and transmission coefficients of two optical potentials $V_1(x)$ and $V_2(x)$ that are obtained from each other by a complex shift of the spatial variable $x$.[18,19] Let

$$\hat{H}_1 \psi(x) = E\psi(x), \tag{16}$$

with $\hat{H}_1 = -d^2/dx^2 + V_1(x)$, $E = k_x^2$, and the optical potential $V_1(x) \to 0$ as $x \to \pm\infty$. For left-side incidence, the wave $\psi(x)$ has the following behavior at infinity that:

$$\psi(x) = \begin{cases} e^{\mathrm{i}k_x x} + r_1^{(L)}(k_x)e^{-\mathrm{i}k_x x} & x \to -\infty \\ t_1(k_x)e^{\mathrm{i}k_x x} & x \to \infty. \end{cases} \tag{17}$$

Now suppose we shift the argument of the potential by an imaginary "distance" so that $x \to x - \mathrm{i}\alpha$ (with $\alpha > 0$). We will also assume that as a function of the complex position $x = x_1 + \mathrm{i}x_2$, $V_1(x)$ does not have any non-analytic behavior (e.g., poles or branch points) in the strip encompassing $0 \leq \mathrm{Im}[x] \leq \alpha$. Let us now consider the Hamiltonian $\hat{H}_2 = -d^2/dx^2 + V_2(x)$ with the optical potential

$$V_2(x) = V_1(x - \mathrm{i}\alpha), \tag{18}$$

which is just the potential $V_1(x)$ shifted by the complex "distance" $\mathrm{i}\alpha$. A solution to the Schrödinger equation for the Hamiltonian $\hat{H}_2$,

$$\hat{H}_2 \phi(x) = E\phi(x), \tag{19}$$

is evidently given by the function $\psi$ appearing in Eq. (16), but with the argument also shifted by a complex distance,

$$\phi(x) = \psi(x - \mathrm{i}\alpha), \tag{20}$$

with the same eigenvalue $E = k_x^2$. Given this simple relationship between $\psi$ and $\phi$, one can relate the reflection from the potential $V_2$ to that from $V_1$ in a rather simple way. Because of Eq. (20) we can write down $\phi(x)$ from Eq. (17) as

$$\phi(x) = \begin{cases} e^{k_x\alpha}e^{ik_x x} + r_1^{(L)}(k_x)e^{-k_x\alpha}e^{-ik_x x} & x \to -\infty \\ t_1(k_x)e^{k_x\alpha}e^{ik_x x} & x \to \infty. \end{cases} \tag{21}$$

Therefore, the transmission and reflection coefficients of the potential $V_2(x)$ are simply obtained from those of the potential $V_1(x)$ via the relations

$$r_2^{(L)}(k_x) = r_1^{(L)}(k_x)e^{-2k_x\alpha} \tag{22}$$

and

$$t_2(k_x) = t_1(k_x). \tag{23}$$

The same relations hold for right-side incidence, provided that the sign of $\alpha$ in Eq. (22) is reversed, giving

$$r_2^{(R)}(k_x) = r_1^{(R)}(k_x)e^{2k_x\alpha}. \tag{24}$$

The relations in Eqs. (22)–(24) show that given some spatially varying permittivity $\epsilon(x)$ (or a potential in the Schrödinger equation), defined for complex arguments $x$, a displacement of this profile by an imaginary "distance" will diminish the reflection of waves incident on one side and enhance the reflection for incidence on the other side. Even if the profile is initially real valued, it will not be after the displacement. The displacement will induce an inhomogeneous distribution of loss and gain necessary to reduce the reflection from one side and enhance it from the other.[20] Note that the transformations of Eqs. (22)–(24) are complex versions of the usual phase shift $e^{2ik_x d}$ one gets in the reflection coefficient after translation of a profile by a real distance $d$, due to the wave having to travel an additional distance $2d$.

## IV. ONE-WAY INVISIBILITY IN KRAMERS-KRONIG OPTICAL POTENTIALS

Here, we demonstrate the existence of a family of one-way invisible permittivity profiles (or potentials), based on the change in the reflection coefficients after a complex displacement of Eqs. (22)–(24) discussed in Sec. III. Let us consider a complex potential $V_2(x)$, related to the permittivity by Eq. (12), with a one-sided Fourier spectrum,

$$V_2(x) = \int_0^\infty dk\, F_2(k) e^{ikx} = \int_0^\infty dk\, F_2(k) e^{ikx_1} e^{-kx_2}, \quad (25)$$

which is a potential constructed out of only right-going waves. Such potentials are always complex valued, and have rather nice behavior in one half of the complex plane. It is clear from Eq. (25) that as one increases $x_2$ from zero, the value of the integral does not become singular for any value of $x_1$ (assuming it was not singular for $x_2 = 0$), and simply becomes a smoother function of $x_1$, continuously reducing in magnitude with increasing $x_2$. Meanwhile, for negative values of $x_2$ the result of the integral can clearly develop singularities, as the exponent on the far right serves to amplify the large $k$ part of the integrand. Figure 2 illustrates this behavior for a particular choice of function $V_2(x)$.

It turns out that whatever one chooses for $F_2(k)$, provided the integral in Eq. (25) converges, the potential $V_2(x)$ is reflectionless for left incidence, and when $F_2(0) = 0$ the potential is also perfectly transmitting, with $t_2(k_x) = 1$. To show this we consider the potential $V_1(x)$ defined as an imaginary shift of $V_2$

$$V_1(x) = V_2(x + i\alpha), \quad (26)$$

with $\alpha > 0$. Such an imaginary displacement pushes all the possible singular behavior of $V_2(x)$ further down into the lower complex position plane, simultaneously reducing the magnitude of the function and smoothing out any fine features (see Fig. 2). By construction, one does not encounter any singular or discontinuous behavior of the potential as one continuously displaces the argument of the potential from the real line $V_2(x_1)$ to some point above $V_2(x_1 + i\alpha)$. Hence the transmission $t_2(k_x)$ and reflection $r_2^{(L,R)}(k_x)$ coefficients of the potential $V_2(x)$ can be related to those of $V_1$ by Eqs. (22)–(24), which were derived in Sec. III,

$$r_2^{(L)}(k_x) = r_1^{(L)}(k_x) e^{-2\alpha k_x}, \quad (27)$$

$$r_2^{(R)}(k_x) = r_1^{(R)}(k_x) e^{2\alpha k_x}, \quad (28)$$

$$t_2(k_x) = t_1(k_x), \quad (29)$$

and which, after comparison with the example in Fig. 2, look interesting. Equation (27) shows that for incidence from the left, the reflection from potential $V_2$ (lower right panel in the figure) is exponentially *smaller* than that from $V_1$ (upper right panel). But $V_1$ varies in space much more slowly than $V_2$, and also exhibits a much lower contrast. This counterintuitive result is avoided if both reflection coefficients are zero, which is indeed the case. To show this, we return to the Fourier integral representation of $V_2(x)$ in Eq. (25), finding that the displaced potential (26) is

$$V_1(x) = \int_0^\infty dk F_2(k) e^{(ikx - \alpha k)}. \quad (30)$$

Let us now consider the behavior of the potential $V_1(x)$ in the extreme limit $\alpha \to +\infty$ (pushing the dashed line in Fig. 2 up to an infinite height). In such a limit, the potential will evidently tend to a very small value. We can find the way in which it tends to this small value by expanding $F(k)$ around $k = 0$, considering the leading-order term of the expansion, $F(k) \sim (A/n!)k^n$, with $A$ constant and $n \geq 1$ [we assume $F(0) = 0$]. Hence in the large $\alpha$ limit we have

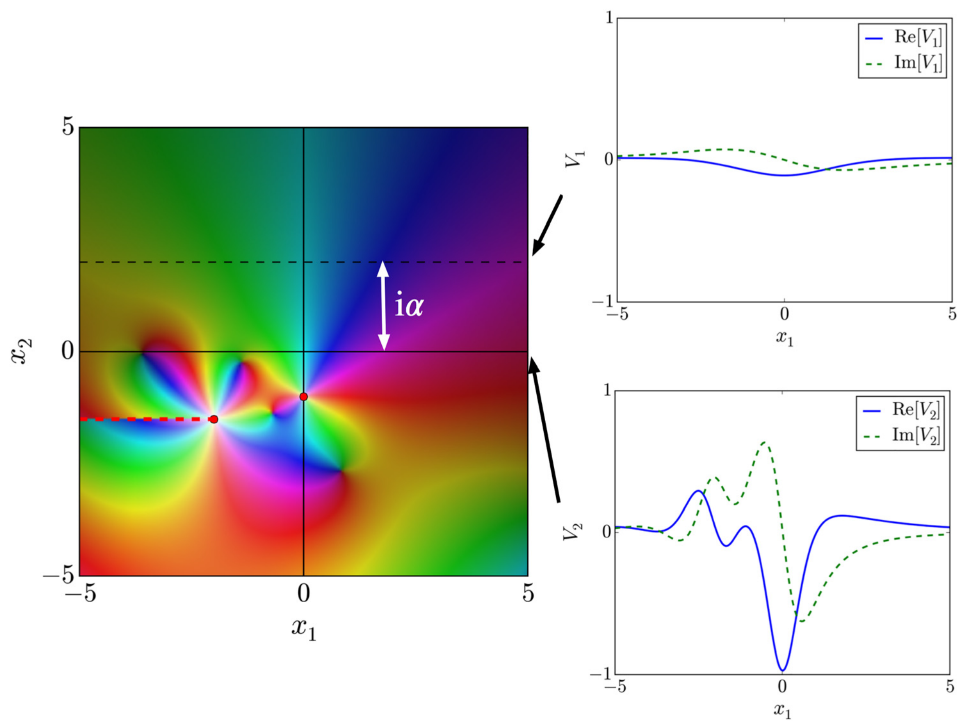


Fig. 2. (Color online) The behavior of a particular (arbitrarily chosen) potential $V_2(x)$ of the form of Eq. (25) as a function of complex position $x = x_1 + ix_2$. The left panel shows a plot of $V_2(x) = 1/(x+i)^2 - e^{ix}/(x+2+3i/2)^{7/2}$, which can be written as a one-sided Fourier transform as given in Eq. (25). The singular points of the function are shown by dots, and the lower dashed line indicates a discontinuity. Shading represents the phase of the function and the brightness the magnitude. The two right-hand panels show plots of the functions $V_1(x_1) = V_2(x_1 + i\alpha)$ (upper) and $V_2(x_1)$ (lower), with $\alpha = 2$.

$$V_1(x) \sim \frac{A}{n!}\int_0^\infty dk\, k^n e^{(\mathrm{i}kx - k\alpha)} = \frac{A}{i^n n!}\frac{d^n}{dx^n}\int_0^\infty dk\, e^{(\mathrm{i}kx-k\alpha)}$$
$$= e^{\mathrm{i}(\pi/2)(n+1)}\frac{A}{(x+i\alpha)^{n+1}}. \tag{31}$$

From Eq. (31), it follows that $V_1(x) \to 0$ as $\alpha \to \infty$, uniformly over the entire real axis. In this limit, the reflection and transmission coefficients of such a vanishing potential $V_1(x)$ are $r_1^{(L,R)}(k_x) \to 0$ and $t_1(k_x) \to 1$ (see the Appendix for details). Since $\alpha$ can be taken arbitrarily large, from Eqs. (27) and (29) one thus obtains

$$r_2^{(L)}(k_x) = 0 \quad \text{and} \quad t_2(k_x) = 1 \tag{32}$$

for all $k_x > 0$. This result implies that the class of optical potentials given in the form of $V_2(x)$ [with $F_2(0) = 0$] are all one-way invisible. Note that we cannot conclude that these potentials are invisible for waves incident from the right-hand side. Even though $r_1^{(R)}(k_k)$ vanishes in the large $\alpha$ limit, Eq. (28) yields $0 \times \infty$, which is indeterminate. Although one cannot make any general conclusion about the reflection coefficient $r_2^{(R)}(k_x)$, one can find special cases where the medium is invisible for both sides of incidence.[16] The preceding argument reproduces the findings of Refs. 16–18, but by much simpler means. However, we have not yet established the connection between the above results and the Kramers-Kronig relations, which we shall do before discussing some examples.

For a wide class of linear causal systems, it is well known that the real and imaginary parts of the Fourier transform of the response function are not unrelated but satisfy the Kramers-Kronig relations.[21–23] In optics, the Kramers-Kronig relations generally relate the real and imaginary parts of the complex susceptibility function $\chi = \epsilon - 1$ of a medium in the frequency domain.[24] The origin of this relationship is that the response of the medium (say, the $z$-component of the material polarization, $P_z$) can only depend on the past behavior of the electric field, or

$$P_z(x,t) = \int_0^\infty \chi(\tau)E_z(x, t-\tau)d\tau. \tag{33}$$

Taking the Fourier transform of Eq. (33), one obtains

$$\tilde{P}(x,\omega) = \tilde{\chi}(\omega)E_z(x,\omega), \tag{34}$$

where

$$\tilde{\chi}(\omega) = \int_0^\infty \chi(\tau)\mathrm{e}^{\mathrm{i}\omega\tau}d\tau, \tag{35}$$

which is of precisely the same form as our reflectionless potentials in Eq. (25), but with frequency swapped with space, and time with $k$. Given that $\tau$ is positive in Eq. (35) we can apply the following integral identity:

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega - \omega'}d\omega = \mathrm{i}\pi\mathrm{e}^{\mathrm{i}\omega'\tau} \qquad \tau > 0. \tag{36}$$

Note that the symbol P denotes the principal part of the integral, which in this case involves skipping out the singular region where $\omega \sim \omega'$, and taking the limit as the region is symmetrically reduced to zero width. Applying this integral identity to the definition of the susceptibility in Eq. (35), we find that the real and imaginary parts of $\chi(\omega)$ are related by

$$\mathrm{Re}[\chi(\omega)] = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}[\chi(\omega')]}{\omega'-\omega}d\omega', \tag{37}$$

and

$$\mathrm{Im}[\chi(\omega)] = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}[\chi(\omega')]}{\omega'-\omega}d\omega'. \tag{38}$$

The above equations are the Kramers-Kronig relations, as they are typically applied to the frequency domain response of a material. Given that our potential in Eq. (25) can be written in the same form as Eq. (35), it must also satisfy the Kramers-Kronig relations (but in space rather than frequency), so that

$$\mathrm{Re}[V(x)] = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}[\mathrm{V}(\xi)]}{\xi - x}d\xi \tag{39}$$

and

$$\mathrm{Im}[V(x)] = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}[\mathrm{V}(\xi)]}{\xi - x}d\xi. \tag{40}$$

We thus reach the same conclusion as in Ref. 16, namely, that permittivity profiles (or potentials) satisfying the spatial Kramers-Kronig relations form a rather broad class of reflectionless materials. In addition, it was shown[17–19] that potentials satisfying the "cancellation condition"

$$\int_{-\infty}^{\infty} dx V(x) = 0, \tag{41}$$

are one-way invisible.[17–19] The condition in Eq. (41) is equivalent to our requirement that the function $F(k)$ in Eq. (25) is zero at $k = 0$: $F(0) = 0$. This can be easily seen from the inverse of Eq. (25), which is

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(x)\mathrm{e}^{-\mathrm{i}kx}dx \to F(0)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(x)\, dx = 0.$$

This cancellation condition also ensures that, far from the inhomogeneity, the scattered waves are plane waves,[17] as assumed in Eqs. (13) and (14). A consequence of the cancellation condition is that the optical medium cannot be purely dissipative (meaning it alternates spatial regions of loss and gain in a balanced manner).

## V. EXAMPLES

### A. Complex shift of a Gaussian potential

The method of complex spatial shift can also be directly used to synthesize one-way invisible media from some starting potential. As an example, let us consider the Gaussian potential barrier ($V_0 > 0$) or well ($V_0 < 0$)

$$V_1(x) = V_0\, e^{-x^2/w^2}, \tag{42}$$

with $V_0$ and $w$ real. The shifted potential $V_2(x) = V_1(x - \mathrm{i}\alpha)$ reads

$$V_2(x) = He^{-x^2/w^2} e^{2\mathrm{i}\alpha x/w^2}, \tag{43}$$

where we have set

$$H = V_0\, e^{\alpha^2/w^2}. \tag{44}$$

The relationship between the reflection coefficients of these two potentials is given in Eq. (22), giving $r_2^{(L)}(k_x) = r_1^{(L)}(k_x)e^{-2k_x\alpha}$, so that the reflection from $V_2$ is exponentially smaller than that from $V_1$.

Let us now consider the limit of a very broad Gaussian potential $V_1$, with $w \to \infty$, and a correspondingly large imaginary displacement $\alpha = aw^2$ (with $a$ assumed order unity) such that the oscillation of the potential $V_2$ does not diverge or tend to zero. In such a limit, the prefactor $H$ of potential $V_2$, defined in Eq. (44), tends to $H = V_0\, e^{a^2w^2}$, which is exponentially larger than $V_0$. We therefore let the amplitude $V_0$ tend to zero as $V_0 = Ae^{-a^2w^2}$ (with $A$ assumed order unity), which in the limit $w \to \infty$ leaves us with the two potentials

$$V_1(x) \to 0 \tag{45}$$

and

$$V_2(x) = \lim_{w\to\infty} Ae^{-x^2/w^2} e^{2\mathrm{i}ax}. \tag{46}$$

In such a limit, the potential $V_2(x)$ reduces to the complex oscillating potential $e^{-2\mathrm{i}\alpha x}$, enveloped by a very broad Gaussian function (see Fig. 3), whereas the potential $V_1(x)$ uniformly vanishes over the entire $x$-axis. Using the aforementioned relationship between the reflection and transmission coefficients of these two potentials, we find (because $V_1$ has a reflection coefficient that approaches zero and a transmission of unity) that

$$r_2^{(L)}(k_x) \to 0 \tag{47}$$

and

$$t_2(k_x) \to 1, \tag{48}$$

meaning the complex potential enveloped by a Gaussian is unidirectionally invisible. This behavior is numerically illustrated in Fig. 3.

### B. Propagating bound states

The bound states of a potential well are formed from a coherent sum of waves reflected from the sides of the well. For a real potential, the sum of these waves is evenly balanced between left and right propagation, indicating that the state remains permanently in the well. The bound state is thus a real function of position and oscillates with a real frequency, without decay over time. The analogous wave in a complex potential will usually not be a bound state (typically exhibiting a complex frequency), and will have energy emanating out of regions of gain or disappearing into dissipative regions. Although we are taught that bound states do not exhibit energy flux, we emphasize that this is only true for real valued potentials. As we shall show, Hamiltonians containing complex potentials can exhibit eigenfunctions (with real eigenvalues) that do not decay or grow over time and with constant energy flow, with the amplification and dissipation of the wave being in perfect balance. An example that is similar to the one presented below can be found in the recent paper of Barashenkov *et al.*[25]

It is plain from the above results, however, that a complex displacement of a real potential changes the balance of the two counter-propagating waves in the bound state according to Eq. (22) without changing the frequency of the wave. This gives rise to the interesting situation in which one can derive potentials that have states with a stable movement of energy from the amplifying to the dissipative regions of the potential, and no overall amplitude change over time. Consider, for example, the harmonic potential

$$V_1(x) = V_0(x/s)^2, \tag{49}$$

where $s$ is a constant with dimensions of length. The eigenstates of $\hat{H}_1$ in Eq. (16) are given by

$$\psi_n(x) = H_n\left(\frac{V_0^{1/4}}{\sqrt{s}}x\right)e^{-\sqrt{V_0}x^2/2s}, \tag{50}$$

where the $H_n$ are Hermite polynomials, and the energies are given by $E_n = 2\sqrt{V_0}s^{-1}(n+1/2)$. For example, the $n = 0$

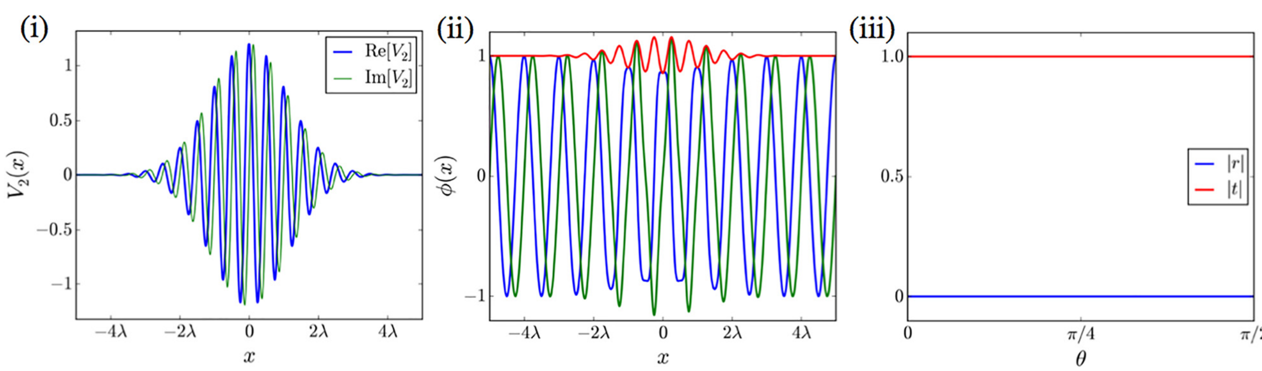


Fig. 3. Illustration of a unidirectionally invisible potential constructed from the complex displacement of a Gaussian ($\lambda = 2\pi/k_0$). (i) Real and imaginary parts of the potential $V_2(x) = k_0^2[\epsilon_b - \epsilon(x)]$ given in Eq. (46), for the arbitrarily chosen values of $A = 1.2$, $a = 2\pi/\lambda$ and $w = 10\lambda/2\pi$. (ii) Numerically calculated behavior of a wave $\phi(x)$ incident from the left onto such a profile; the real (dark, blue) and imaginary (light, green) parts oscillate about 0, while the absolute value (upper, red) oscillates about 1. (iii) Magnitude of reflection (blue) and transmission (red) coefficients from the potential shown in panel (i) as a function of $\theta$ [$k_y = k_0\cos(\theta)$].

ground state is simply a Gaussian, which is an equal superposition of left and right propagating waves

$$\psi_0(x) = e^{-\sqrt{V_0}x^2/2s} = \frac{\sqrt{s}}{V_0^{1/4}}\int_0^\infty \frac{dk}{\sqrt{2\pi}} e^{-sk^2/2\sqrt{V_0}}\left(e^{ikx}+e^{-ikx}\right). \tag{51}$$

The displaced potential $V_2(x) = V_1(x - i\alpha)$ is given by

$$V_2(x) = \frac{V_0}{s^2}\left(x^2 - 2i\alpha x - \alpha^2\right), \tag{52}$$

which, relative to $V_1(x)$, is shifted down by $V_0\alpha^2/s^2$ and has an additional linear imaginary part that exhibits dissipation where $x > 0$ and gain where $x < 0$ ($\alpha > 0$). The analogous state of the Hamiltonian $\hat{H}_2$ in Eq. (19) is

$$\phi_0(x) = \frac{\sqrt{s}}{V_0^{1/4}}\int_0^\infty \frac{dk}{\sqrt{2\pi}} e^{-sk^2/2\sqrt{V_0}}\left(e^{ikx}+e^{-ikx}e^{-2k\alpha}\right), \tag{53}$$

which oscillates with a real frequency (the same as $\psi_0$), but has a predominance of right-going waves compared to left-going ones. For large $\alpha$ the state is stable, but has power continuously flowing from left to right (Fig. 4).

## VI. CONCLUSIONS

We have outlined a method where, through considering complex "spatial" displacements of a permittivity profile (or potential), we can derive results for wave reflection and transmission without ever needing the exact solution to the wave equation. Although we derived this for electromagnetic waves of a fixed polarization, the same procedure can be carried out for both polarizations with only a change to the form of the optical potential of Eq. (12).[16] Moreover, the results are certainly not constrained to the context of electromagnetic waves, but hold for any wave so long as there is a clear way to implement both loss and gain (e.g., absorbers and amplifiers are common in acoustics). The results from this method are also not constrained to the case of unidirectional invisibility, and in some cases bidirectional invisibility is also possible.[26] Through applying this to potentials that are analytic in one half of the complex plane, we demonstrated in a rather simple way some of the relevant properties of reflection in isotropic dielectric media shown in the recent work,[16–18] including a derivation of a very wide class of one-way invisible potentials. In two examples, we showed how the method can also be applied to derive new one-way invisible potentials that fall outside the class of potentials discussed,[16–18] and that stable "bound states" of complex potentials can be similarly derived.

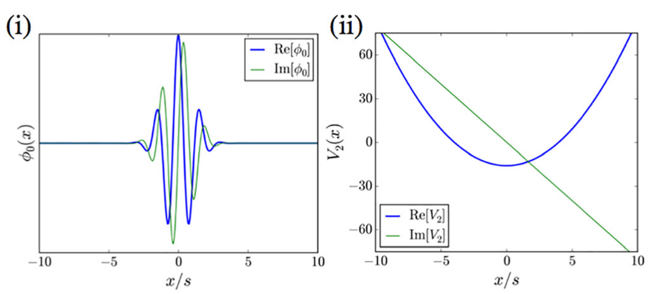


Fig. 4. Complex shift applied to the bound states of a potential well, yielding a $\mathcal{PT}$-symmetric potential with stable states where there is a continuous propagation of energy from one side of the well to the other. (i) Stable state $\phi_0(x)$ of Eq. (53) for $\alpha = 4s$ showing a clear advance in phase, demonstrating propagation of the wave from the left to the right of the well. (ii) Complex potential, from Eq. (52), which amplifies the wave for $x < 0$, and absorbs it for $x > 0$.


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## APPENDIX: REFLECTION AND TRANSMISSION OF A VANISHING POTENTIAL

Let us consider the Schrödinger-like wave equation with a scattering potential $V(x)$ of small amplitude (order $\eta \ll 1$)

$$-\frac{d^2\psi}{dx^2} + \eta V(x)\psi = E\psi, \tag{A1}$$

where $V(x)$ is of order unity, $E = k_x^2$, and $V(x) \to 0$ as $x \to \pm\infty$ with $\int_{-\infty}^{\infty} V(x)\,dx = 0$. As is done in the Born approximation,[27] we look for a solution to Eq. (A1) as a power series in $\eta$, so that

$$\psi(x) = \psi^{(0)}(x) + \eta\psi^{(1)}(x) + \ldots. \tag{A2}$$

To leading order in $\eta$ one has

$$\frac{d^2\psi^{(0)}}{dx^2} + k_x^2\psi^{(0)} = 0, \tag{A3}$$

with a solution, corresponding to a traveling wave coming from the left, given by

$$\psi^{(0)}(x) = e^{ik_x x}. \tag{A4}$$

At the next order in $\eta$ one has

$$\frac{d^2\psi^{(1)}}{dx^2} + k_x^2\psi^{(1)} = V(x)\psi^{(0)}(x) = V(x)e^{ik_x x}, \tag{A5}$$

which is the differential equation of the forced harmonic oscillator. One solution to such an equation, which does not change the amplitude of the incoming wave from the left, is given as a superposition of out-going waves from the region where the potential is significantly different from zero, or

$$\psi^{(1)}(x) = \int_{-\infty}^{\infty} \frac{e^{ik_x|x-x'|}}{2ik_x} V(x')e^{ik_x x'}\,dx', \tag{A6}$$

as one can readily check by direct substitution into Eq. (A5). Note that $\psi^{(1)}(x)$ can be alternatively cast in the form

$$\psi^{(1)}(x) = \frac{e^{ik_x x}}{2ik_x}\int_{-\infty}^{x} V(x')\,dx' + \frac{e^{-ik_x x}}{2ik_x}\int_{x}^{\infty} V(x')e^{2ik_x x'}\,dx'. \tag{A7}$$

If we limit ourselves to stopping at order $\eta$ and take into account the "cancellation condition" $\int_{-\infty}^{\infty} V(x')\,dx' = 0$, we find reflection and transmission coefficients equal to

$$r^{(L)}(k_x) \sim \frac{\eta}{2\mathrm{i}k_x}\int_{-\infty}^{\infty} V(x')e^{2\mathrm{i}k_x x'}\,dx' \tag{A8}$$

and

$$t(k_x) \sim 1. \tag{A9}$$

Hence, $r^{(L)}(k_x) \to 0$ and $t(k_x) \to 1$ as $\eta \to 0$ for any $k_x > 0$ [the integral in Eq. (A8) converges]. In a similar way, one can show that $r^{(R)}(k_x) \to 0$ as $\eta \to 0$.

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